Time-dependent Convection on Self-Adaptive Grids

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Abstract

A method for calculating energy transfer due to turbulent convection in self-gravitating radiating spherical flows of general, non-ideal media is decribed. A model for time-dependent turbulent convection in stars due to Kuhfuß [12, 13], in its simplest version is discretized on a self-adaptive grid.

1 Convective Energy Transfer

Convection is a notorious problem in astrophysics. In most astrophysical problems the equations to be solved to get insight and understanding are well known and answering an astrophysical question is largely applied physics. Quite the contrary for convection. A key concept, the Schwarzschild criterion originated from an astrophysical question [14], namely the structure of the solar photosphere and the equations to be solved to determine even the most basic effects like energy transfer are yet under discussion. For a review of the various theoretical approaches to convection see Baker [1] and, in the solar context Spruit et al. [15].

On the other hand it is a necessity to include convection in many branches of astrophysics. Therefore modelers confronted with the lack of a theory of convection have always developed convection prescriptions with properties that summarized the available theoretical and empirical knowledge [1]. Free parameters are elimited by the use of properties of certain model systems. The parameterisation of the mixing lenght e.g. is determined from the age of the solar system as measured from meteorites.

With this in mind Kuhfuß' [12, 13]) formulation offers the following advantages as compared to zero-entropy-gradient or mixing-lenght prescriptions:

- The model is developed in the context of hydrodynamics without phenomenological concepts.
- In this framework the assumptions made have a clear meaning and their consequences can be followed in the derivation of the model equations.
- The model has relatively few (three in the simplest time-dependent version) free parameters.
- The uncertainties have been put into positions that allow a physically meaningful definition of the free parameters.
- For a standard selection of the free prameters, the simplest version (almost) exactly reduces to mixing length theory.
- For simple cases a number of analytical properties of the model, have been derived, [13].
- Inclusion of various effects, like overshooting, turbulent viscosity, radiative losses of eddies, ..., have been developed and are straightforward extensions of the simplest version.

For further discussion and comparison to other convection models see [12, 13, 5, 6]. The remaining task was to formulate the model in a way that (1) remains mathematically well behaved, in flows that are far from hydrostatic equilibrium and (2) numerically well behaved in the context of implicit self-adaptive mesh methods in radiation hydrodynamics.

How static zero-entropy-gradient convection could be included in radiation-hydrodynamical calculations on a self-adaptive mesh, was demonstrated by Wuchterl [21, 22]. The flows calculated in the context of giant planet formation involve strong shock waves, accretion and winds. Furthermore the question arose whether time-dependent convection would effect the excitation mechanism for protoplanetary pulsations. That reemphasized the need for a time dependent formulation of convection, a demand that previously came from star-formation and stellar pulsation studies. To estimate the effect of time dependent convection on the protoplanetary excitation mechanism we developed a formulation and discretisation of Kuhfuß′ [13] model. The first published calculations with a slighly different version of the formulation described in this article relate to slowly rotating protoplanets [7].

In the following we will discuss how to reformulate and discretize the simplest version of Kuhfuß' model for convection to include convective energy transfer in self-gravitating radiation gas-dynamics on self-adaptive grids.

2 Kuhfuß 1987 Model for Turbulent Convection

In his 'one-equation-model' Kuhfuß [12, 13] derives an equation for the specific turbulent kinetic energy density $\tilde{\omega}$ which in the simplest approximation (turbulent pressure and stresses as well as diffusion of turbulent kinetic energy ('overshooting') are neglected) reads:

$$\frac{d\tilde{\omega}}{dt} = S_{\tilde{\omega}} - \hat{S}_{\tilde{\omega}}. \tag{1}$$

The total time derivative of $\tilde{\omega}$ is determined by excitation and damping of convective motion, $S_{\tilde{\omega}}$ due to the bouyancy-force and dissipation of $\tilde{\omega}$ into smaller, ultimately molecular scales $\hat{S}_{\tilde{\omega}}$. The square-root of $\tilde{\omega}$ is the typical velocity of a convective element. Dimensional analysis leads Kuhfuß to:

$$\hat{S}_{\tilde{\omega}} = c_D \frac{\tilde{\omega}^{3/2}}{\Lambda}. \tag{2}$$

Neglecting pressure fluctuations, i.e. assuming pressure equilibrium between convective elements and their environment, Kuhfuß finds for the bouyancy-term of a chemically homogenous medium:

$$S_{\tilde{\omega}} = \frac{1}{\varrho^2}\frac{\partial p}{\partial r}\left(\frac{\partial \varrho}{\partial s}\right)_p \Pi. \tag{3}$$

The enthalpy-flux j_w caused by the turbulent motion of the convective elements is

$$j_w = \varrho T \Pi, \tag{4}$$

and an approximation for the correlations between velocity- and entropy-fluctuations

$$\Pi = -\alpha_S \Lambda \tilde{\omega}^{1/2}\left(\frac{\partial s}{\partial r}\right) \tag{5}$$

closes the system. Creation and destruction of $\tilde{\omega}$ as well as energy transfer by j_w have to be budgeted in the thermal energy equation. This leads to coupling between the turbulent elements ('eddy-gas') and the gas. Momentum coupling via the turbulent pressure is neglected here but included in the Kuhfuß description. The internal energy equation (first law of thermodynamics) including the internal energy of the eddy gas, $\tilde{\omega}$ than reads:

$$\frac{d}{dt}(e+\tilde{\omega}) + \frac{P}{\varrho}\mathrm{div}u = -\frac{1}{\varrho}\mathrm{div}(j_w + F_{\mathrm{rad}}). \tag{6}$$

This description of turbulent convection reduces to the mixing length theory for steady solutions and a particular choice of the parameters α_S and c_D, i.e.

$$\alpha_S = \frac{1}{2}\sqrt{\frac{2}{3}} \tag{7}$$

$$c_D = \frac{8}{3}\sqrt{\frac{2}{3}} \tag{8}$$

Convective velocity and convective flux i.e. $u_c = \sqrt{2/3\tilde{\omega}}$ and j_w, respectively are then and only then equal to values derived from the mixing length theory, cf. [2, 3, 9]. The corresponding overadiabaticity, $\nabla - \nabla_s$, however, is somewhat different (always within a factor 1.8) when calculated from the mixing length theory. This is due to the fact that radiative losses of the rising convective elements have been neglected in Kuhfuß' 'one-equation-model'. To arrive at mixing length theory-values for the super-adiabaticity one could replace ∇_s by the MLT's ∇' in the source term, $\hat{S}_{\tilde{\omega}}$ and note that

$$-\left(\frac{\partial s}{\partial r}\right) = \frac{c_P}{H_P}(\nabla - \nabla_s). \tag{9}$$

A non ad hoc but much more complicated way to determine the radiative losses would be to use Kuhfuß' 'three equation model'. For the details of the discussion see Kuhfuß' thesis [13].

3 Reformulation for Radiation Gas Continuum Dynamics

Kuhfuß' uses derivatives of the specific entropy to develop his convection model. For application in quasi-hydrostatic stellar structure and evolution calculations and comparisons to mixing lenght theory and other convection models it is useful to reformulate the model in terms of the so called ∇'s, i.e. temperature gradients with respect to pressure that are widely used in the theory of stellar structure, cf. [13, 10]. However this is no appropriate way to proceed towards a formulation for application in hydrodynamic flows where the pressure, in contrast to hydrostatic equilibria cannot be used as a coordinate and the ∇'s become singular wherever $\partial P/\partial r$ vanishes, e.g. in the rarefaction region behind an upward moving mass shell. Thus our starting point is Kuhfuß 'entropy gradient formulation' that has been briefly summarized in the preceeding section 2.

Following Winkler and Norman [20], and Tscharnuter [16] we use density and internal energy as independent variables for accuracy of the numerical radiation hydrodynamics. Using the first two laws of thermodynamics we rewrite the entropy gradient in those variables. Next we are confronted with the need for an appropriate generalisation of the parametrisation of the mixing length in terms of the local pressure scale height $-\partial r/\partial \ln P$, which apparently suffers from the same singularity for $dP = 0$ as the ∇'s. To obtain the correct limit for hydrostatic stratifications without introducing new assumptions, the mixing lenght is written in the *hydrostatic pressure scale height*; i.e. the pressure scale height of a hydrostatic equilibrium corresponding to the values of pressure P and density ϱ in the actual, in general dynamic flow:

$$H_P^{\mathrm{stat}} = - \left. \frac{dr}{d\ln P} \right|_{\mathrm{hydrostatic}} = \frac{r^2}{GM_r} \frac{P}{\varrho} \tag{10}$$

Note that this is only a reasonable procedure for the parametrisation of the mixing length and must not be used to transform temperature gradients with respect to pressure along the flow into spatial gradients when converting ∇-formulations of convection models. Further complications arise under conditions of weak gravity, where $H_P^{\mathrm{stat}}/r > 1$. (Such conditions have e.g. been encountered in the extended envelopes of protostars and proto-giant planets). We assume that the local radius r is an appropriate lenghtscale for measuring the largest scale turbulent motions and adopt a harmonic mean:

$$\frac{1}{\Lambda} = \frac{1}{\alpha_{\mathrm{ML}} H_P^{\mathrm{stat}}} + \frac{1}{\beta_r r} \tag{11}$$

as a parametrisation for the mixing length that is physically reasonable under all conditions and reduces to the standard mixing length prescription under 'standard' hydrostatic conditions with $H_P^{\mathrm{stat}}/r < 1$. The standard value for $\alpha_{\mathrm{ML}} = 1.5$ and $\beta_r = 0.1$ was a typial value used in protoplanetary models by the author. Note that H_P^{stat}/r essentially is the ratio of thermal to gravitational energy and is a dimensionless measure for the flatness of a hydrostatic stratification. It sets a lower limit for the required spatial resolution for the study of such structures and is a good indicater for the numerical toughness of a problem in self-gravitating radiation hydrodynamics. On that scale proto-giant planet are easiest, followed by protostellar cores, RR Lyrae and finally Cepheïd stars, to mention a few applications.

4 Reformulation for Implicit Self-Adaptive Methods

A key problem in solving eq. 1 for the turbulent kinetic energy is that there exist two branches of solutions. To illustrate this we look at the explicit form of Equ. 1 in the static case. Setting $d/dt = 0$ and substituting $S_{\tilde{\omega}}$, $\hat{S}_{\tilde{\omega}}$ and Π we obtain:

$$-\alpha_S \Lambda \tilde{\omega}^{1/2} \left(\frac{\partial s}{\partial r} \right) \frac{1}{\varrho^2} \frac{\partial p}{\partial r} \left(\frac{\partial \varrho}{\partial s} \right)_p - c_D \frac{\tilde{\omega}^{3/2}}{\Lambda} = 0 \tag{12}$$

This is a cubic equation for $\tilde{\omega}^{1/2}$ of the form

$$\tilde{\omega}^{1/2}(A - \tilde{\omega}) = 0 \tag{13}$$

with

$$A = \frac{\alpha_S}{c_D}\Lambda^2 \frac{T}{P}\nabla_s \frac{\partial p}{\partial r}\left(\frac{\partial s}{\partial r}\right) \tag{14}$$

The physically interesting branch in convectively stable regions ($A < 0$) is $\tilde{\omega} = 0$. For $A > 0$ convective instability is indicated and the two solutions of the second branch become real. Then the positive value of the second branch is the desired value for the turbulent kinetic energy in a convective region.

The numerical strategy to solve the general time-dependent equation for $\tilde{\omega}$ is to make use of this property. In stable stratifications the solution $\tilde{\omega} = 0$ can be readily found by the Newton-algorithm. In convectively unstable regions the second branch becomes real, but $\tilde{\omega} = 0$ remains a solution. To obtain the physically interesting finite solution the guess required for the Newton-algorithm is set to a large enough value that it is 'closer' to the second branch. A simple choice for this guess is to use the algebraic solution of the static turbulent kinetic energy equation, i.e. $\tilde{\omega}^{1/2} = \sqrt{A}$.

The iterative solution of the implicit difference equations by a Newton-Raphson-Algorithm requires the evaluation of the corresponding Jacobian. For convectively stable stratification $\tilde{\omega} = 0$ and $\partial_{\tilde{\omega}} S_{\tilde{\omega}}$ and $\partial_{\tilde{\omega}} \hat{S}_{\tilde{\omega}}$ are singular. It has been suggested that a way around this is to 'seed' equation 1, i.e. at every time level to set a small but finite value for $\tilde{\omega}$ and use the equation to amplify or damp that seed of turbulence. In actual calculations the appropriate choice of the seed turns out to be difficult, the growth strongly depends on the seed [13], and the singularity remains in the Jacobian.

We suggest to use the *convective velocity* $u_c = \sqrt{\frac{2}{3}\tilde{\omega}}$ as independent convective variable. That leads to a regular Jacobian for $\tilde{\omega} = 0$.

5 Discretisation

We follow the rules of discretisation on a staggered mesh as described by Tscharnuter [17], and Winkler and Norman [20]. Scalars and scalar equations are defined in the cell interiors, i.e. between the gridpoints whereas vector and vector equations are defined on the boundaries (gridpoints in 1D). The aim is to discretize the model for turbulent convection in the framework of an implicit, first order accurate scheme with donor cell advection on a self-adaptive mesh. A recent overiew of this technique is given by Tscharnuter [17]. Discretisation details can be obtained in [21, 20]. For the self-adaptive grid equation we refer to Dorfi and Drury [4]. A second order accurate implementation of radiation hydrodynamics with the convection model discussed here and van Leer-advection has recently been implemented in Vienna by Feuchtiger and Wuchterl (priv. comm.).

The first implementation of the discretisation was in an existing radiation-hydrodynamics code used for giant planet formation by Wuchterl [21, 22]. For technical reasons of that code the 'core'-matrix of the Henyey-method (cf. [8, 11]) required a restriction to a three point stencil. This property of the discretisation may still be used for computational advantage.

As far as the staggered mesh is concerned the eq. 1 for the turbulent kinetic energy is of a mixed type. It incorporates the continuity equation, the equation of motion and the energy equation of the turbulent elements (eddies). It is written in $\tilde{\omega}$ which is a scalar energy density essentially determining the vector convective velocity $u_c = \sqrt{\frac{2}{3}\tilde{\omega}}$.

Looking at the all important coupling to the energy equation via j_w, we arrive at $\left(\frac{\partial s}{\partial r}\right)$, a gradient, as the guiding object for discretisation. The following discrete versions of eq. 1, and the coupling terms of the 'eddy-gas' with matter in the energy equation are written using the following notation:

$$\text{Time level difference} \qquad \delta x = x - x^* \tag{15}$$

$$\text{Spatial difference} \qquad \Delta x_j = x_j - x_{j+1} \tag{16}$$

$$\text{Volume difference} \qquad \Delta \mathrm{Vol}_j = \frac{1}{3}\Delta r_j (r_j^2 + r_j r_{j+1} + r_{j+1}^2) \tag{17}$$

$$\text{Turbulent kinetic energy} \qquad \tilde{\omega}_j = \frac{3}{2} u_{c,j}^2 \tag{18}$$

$$\text{Donnor-cell advection} \qquad f_j^{\mathrm{adv}}(\dot{\xi}, x) = \begin{cases} -x_{j-1}\dot{\xi}_j & \dot{\xi}_j < 0 \\ 0 & \dot{\xi}_j = 0 \\ -x_j \dot{\xi}_j & \dot{\xi}_j > 0 \end{cases} \tag{19}$$

ξ is the mass M_r interior to radius r, divided by 4π, and $\dot{\xi}$ its time derivative with respect to the grid. Note that the volume as well as the turbulent kinetic energy and internal energy equation below have been divided by 4π.

6 The Equation for the Turbulent Kinetic Energy

The discretized Version of eq. 1 is displayed in Tab. 1. The functions $P(\varrho, e)$, $T(\varrho, e)$ and $\nabla_s(\varrho, e)$ are interpolated from equation of state tables.

7 Guessing the Static Solution

The numerical strategy outlined above details as follows. Before every iteration u_c is limited by zero, i.e. $u_c = \max(u_c, 0)$. To solve the entire discrete radiation

$$\delta(\varrho_j \tilde{\omega}_j \Delta\mathrm{Vol}_j) + \Delta f_j^{\mathrm{adv}}(\dot{\xi}, \tilde{\omega}) - \delta t \varrho_j \Delta\mathrm{Vol}_j (S_{\tilde{\omega},j} - \hat{S}_{\tilde{\omega},j}) = 0 \tag{20}$$

$$\left(\frac{\partial s}{\partial r}\right)_j = \frac{1}{\Delta r_{j-1}} \left(\frac{\Delta e_{j-1}}{T_j} - \frac{P_j}{\varrho_j^2} \frac{\Delta \varrho_{j-1}}{T_j} \right) \tag{21}$$

$$\hat{S}_{\tilde{\omega},j} = \frac{c_D}{\Lambda_j} \left(\sqrt{\frac{3}{2}} u_{c,j} \right)^3 \tag{22}$$

$$S_{\tilde{\omega},j} = \alpha_S \Lambda_j \left(\frac{\partial s}{\partial r}\right)_j \frac{\Delta P_{j-1}}{\Delta r_{j-1}} \nabla_{s,j} \sqrt{\frac{3}{2}} u_{c,j} \tag{23}$$

$$\Lambda_j = \left(\frac{1}{\alpha_{\mathrm{ML}} H_{P,j}^{\mathrm{stat}}} + \frac{1}{\beta_r r_j} \right)^{-1} \tag{24}$$

Table 1: Discretized equation for the turbulent kinetic energy, $\tilde{\omega}$ written in terms of the convective velocity, $u_c = \sqrt{\frac{2}{3}\tilde{\omega}}$.

hydrodynamics system a guess is needed. It has to provide values for all flow quantities, including the convective velocity, i.e. u_c^g. If that guess indicates convective instability ($A > 0$) and $u_c^g < \epsilon u_c^0$, the starting value for the iterative solution is set to the static solution of the equation for the convective velocity, i.e.:

$$u_c^0 = \sqrt{\frac{2}{3} A} = \sqrt{\frac{2}{3} \frac{\alpha_S}{c_D} \Lambda_j \left(\frac{\partial s}{\partial r}\right)_j \frac{\Delta P_{j-1}}{\Delta r_{j-1}} \nabla_{s,j} \frac{T_j}{P_j}}\,, \tag{25}$$

where ϵ is a number smaller than the accuracy requirement for the solution of the difference equations; values of 10^{-8} to 10^{-12} have been used in practice.

8 Convection Terms in Internal Energy Equation

The total derivative of $\tilde{\omega}$ in the identical form as in the equation for the turbulent kinetic energy is added to the internal energy equation, and the divergence of the convective enthalpy flux, j_w is discretised in analogy to the respective term for the radiative flux (cf. [20]).

$$\delta(\varrho_j \tilde{\omega}_j \Delta\mathrm{Vol}_j) + \Delta f_j^{\mathrm{adv}}(\dot{\xi}, \tilde{\omega}) + \Delta(r_j^2 j_{w,j}) + \ldots = 0 \tag{26}$$

$$j_{w,j} = -\alpha_S \Lambda_j \varrho_j T_j \sqrt{\frac{3}{2}} u_{c,j} \left(\frac{\partial s}{\partial r}\right)_j \tag{27}$$

9 Boundary Conditions

The following boundary conditions for convection are appropriate for giant planet formation calculations and are given as an example. The inner boundary at $r_{\min}$ is a solid, gravitating core with no convection accross its surface and the protoplanetary envelope is embedded in the solar nebula allowing convection and mass exchange across the outer boundary at $r_{\max}$.

$$r_{j+1} = r_{\min} \qquad j_{w,j+1} = 0 \tag{28}$$

$$f^{\mathrm{adv}}_{j+1} = 0 \tag{29}$$

$$r_j = r_{\max} \qquad u_{c,j-1} = u_{c,j} \tag{30}$$

In the protoplanet-version of the radiation-hydrodynamics program, [21, 22] the vanishing of advection of $\tilde{\omega}$ accross the core surface was realised by putting $u_{c,j_{\max}} = 0$.

10 Discussion

10.1 Explicit Mixing Length as Shortcut

To save some effort in the implementation of the convection equations one may calculate the mixing lenght explicitely, i.e. on the old time-level. This worked well in the protoplanetary case but yielding a smaller convergence-radius for the iterative solution of the difference equations and consequently smaller time-steps. However, if the suberadiabaticty of the stratification in convection zones decreases below $\left(\frac{\partial \ln s}{\partial \ln r}\right)_{\mathrm{strat}} \approx 10^{-7}$ problems arise. With $\left(\frac{\partial s}{\partial r}\right)$ going even closer to zero, sudden divergences of the Newton-Raphson algorithm occur. Typical physical situations for that are convective regions at very high optical depths, making radiative transfer very inefficient and leading to the radiative luminosity which becomes a very small fraction (10^{-7}, say) of the total luminosity. Therefore whenever such situations are expected it is worth to do the additional programming for calculating the mixing lenght implicitly.

10.2 Negative ∇_s

Exotic properties of convective energy transfer, like convective stability for superadiabatic stratifications arise when ∇_s obtains negative values. The isentropic temperature gradient ∇_s may be ad hoc set to a constant value in the source term $S_{\tilde{\omega}}$ to obtain 'normal convection', cf. [7]. Negative values of ∇_s are 'abnormal' but not excluded by basic thermodynamical principles. They occured in the deep convective interiors of proto-giant planets. Pressure dissociation/ionisation under degenerate conditions as treated in an equation of state calculated by Wuchterl, [21] indicate those values. The uncertainties in that domain are substantial and do not validate the additional effort to discuss the complications caused by convection under such special conditions. The simplification in the numerical treatment under those abnormal conditions in protoplanets was substantial and the inaccuracies introduced into the treatment of convection in regions of the flow with $\nabla_s > 0$, are minor. Formally this can be viewed as a modification of α_s that will lead to another small modification of the superadiabaticity relativ to the standard mixing lenght theory.

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